

Large deviations for a fractional stochastic heat equation in spatial dimension \mathbb{R}^d driven by a spatially correlated noise

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Abstract

In this paper we study the Large Deviation Principle (LDP in abbreviation) for a class of Stochastic Partial Differential Equations (SPDEs) in the whole space \mathbb{R}^d , with arbitrary dimension $d \geq 1$, under random influence which is a Gaussian noise, white in time and correlated in space. The differential operator is a fractional derivative operator. We prove a large deviations principle for our equation, using a weak convergence approach based on a variational representation of functionals of infinite-dimensional Brownian motion. This approach reduces the proof of LDP to establishing basic qualitative properties for controlled analogues of the original stochastic system.

Keywords: Fractional derivative operator; stochastic partial differential equation; correlated Gaussian noise; Fourier transform; large deviation principle; weak convergence method.

AMS Subject Classification: 60F10, 60G15, 60H15.

1 Introduction and general framework

In this paper we consider the following Stochastic Partial Differential Equation (SPDE in abbreviation) given by

$$\begin{cases} \frac{\partial u^\varepsilon}{\partial t}(t, x) = \mathcal{D}_{\delta}^{\alpha} u^\varepsilon(t, x) + b(u^\varepsilon(t, x)) + \sqrt{\varepsilon} \sigma(u^\varepsilon(t, x)) \dot{F}(t, x), \\ u^\varepsilon(0, x) = 0, \end{cases} \quad (1.1)$$

where $\varepsilon > 0$, $(t, x) \in [0, T] \times \mathbb{R}^d$, $d \geq 1$, $\alpha = (\alpha_1, \dots, \alpha_d)$, $\delta = (\delta_1, \dots, \delta_d)$ and we will assume, along this paper, that $\alpha_i \in]0, 2] \setminus \{1\}$ and $|\delta_i| \leq \min\{\alpha_i, 2 - \alpha_i\}$, $i = 1, \dots, d$.

\dot{F} is the “formal” derivative of the Gaussian perturbation and $\mathcal{D}_{\delta}^{\alpha}$ denotes a non-local fractional differential operator on \mathbb{R}^d defined by

$$\mathcal{D}_{\delta}^{\alpha} = \sum_{i=1}^d D_{\delta_i}^{\alpha_i},$$

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where $D_{\delta_i}^{\alpha_i}$ denotes the fractional differential derivative w.r.t the i -th coordinate defined via its Fourier transform \mathcal{F} by

$$\mathcal{F}\left(D_{\delta_i}^{\alpha_i}\varphi\right)(\xi)=-|\xi|^{\alpha_i}\exp\left(-i\delta_i\frac{\pi}{2}\text{sgn}(\xi)\right)\mathcal{F}(\varphi)(\xi),$$

where $i^2 + 1 = 0$.

The noise $F(t, x)$ is a martingale measure (in the sense given by Walsh in [38]) to be defined with more details in the sequel. The coefficients b and $\sigma : \mathbb{R} \rightarrow \mathbb{R}$ are given functions. We shall refer to equation (1.1) as $Eg_{\delta, \varepsilon}^{\alpha}(d, b, \sigma)$.

The theory of large deviations, which has been investigated for different systems in recent years, reveals important aspects of asymptotic dynamics. Particular attention has been paid to studying LDP for stochastic differential equations (SDE) (e.g. [1], [35], [10], [30], [22]).

In this contribution to the theory of Large deviations, first investigated by Freidlin and Wentzell in there original work [21] for Brownian noise driven SDEs in finite dimension, we are interested in the study of a stochastic heat equation in spatial dimension $d \geq 1$ driven by a spatially correlated noise using the approach in [21] (see also [17]).

In various papers about the LDP for solutions to SPDEs or to stochastic evolution equations in a semi-linear framework [3, 5, 6, 8, 17, 19, 22, 25, 30], the strategy used is similar to the classical one for diffusion processes and based on standard LDP discretization methods which, for a large number of problems of significant interest, necessitate exponential probability estimates and exponential tightness much hardest and most technical.

There exist two distinct methods, in establishing LDP for an SDE or SPDE with multiplicative noise, the classical approach and the weak convergence method. In the classical method one should discretize the time horizon and freeze the diffusion term on each interval and then use the Varadhan's contraction principle. In this method we should overcome many difficult inequalities for convolution integrals. In the weak convergence approach, which is the approach employed in this work, we should obtain some sort of continuity w.r.t. some control variables. We will clarify further this approach in the current and next section. Several authors have studied LDP for infinite dimensional SDEs with Lévy noise, see [35], [10] for the classical one and [4, 7] for the weak convergence approach.

In this paper we will prove a tantamount argument of the large deviations principle, the Laplace principle, and we will study the Uniform Laplace Principle. The reader should refer to [20] for a proof of the aforementioned equivalence.

In order to prove our LDP result, we use a combination of a variational representation for infinite-dimensional Brownian motion and a transfer principle via Laplace Principle based on compactness and weak convergence proved in [8]. Using this method, an LDP to a reaction-diffusion system has been obtained in [9]. For the case of wave equation in spatial dimension $d = 3$, the same method has been applied to derive an LDP result in [29], and also in several subsequent papers, for instance see [19, 36]. The case of heat equation, governed by the same noise, has been considered in [24] taking the spatial coordinate $x \in [0, 1]^d$, $d \geq 1$ where the authors needs to establish precise estimates of the fundamental solution in order to obtain a Freidlin-Wentzell type inequality. In contrast, the approach we take in this paper is different from that of [24], and it is based on weak convergence arguments. We can also refer to original

LDP result for the case of a one-dimensional heat equation driven by a Brownian sheet in [10].

A summary of the paper is as follows. The first and second subsections will give definitions and preliminary results about the fractional operator and the noise both considered in this paper. Next, we present the set of assumptions required for the existence, uniqueness to equation $Eq_{\delta,\varepsilon}^\alpha(d, b, \sigma)$, integrability condition for the stochastic integral and also regularity properties of the solution. The last section is devised into three subsections.

The first one reviews basic definitions of LDP, the Laplace principle (LP), the equivalence in between in the setting of Polish spaces and a brief description of the method we follow in this work. The statement of the main result along with the set of assumptions to be checked are given in the second subsection. The last one will present the blocks of our proofs. The value of the constants along this article may change from line to line and sometimes we shall emphasize the dependence of these constants upon parameters.

1.1 The operator $\mathcal{D}_\delta^\alpha$

In one space dimension, the operator D_δ^α is a closed, densely defined operator on $L^2(\mathbb{R})$ and it is the infinitesimal generator of a semi-group which is in general not symmetric and not a contraction. It is self adjoint only when $\delta = 0$ and in this case, it coincides with the fractional power of the Laplacian.

According to [18, 23], D_δ^α can be represented for $1 < \alpha < 2$, by

$$D_\delta^\alpha \varphi(x) = \int_{-\infty}^{+\infty} \frac{\varphi(x+y) - \varphi(x) - y\varphi'(x)}{|y|^{1+\alpha}} \left(\kappa_-^\delta \mathbf{1}_{(-\infty, 0)}(y) + \kappa_+^\delta \mathbf{1}_{(0, +\infty)}(y) \right) dy,$$

and for $0 < \alpha < 1$, by

$$D_\delta^\alpha \varphi(x) = \int_{-\infty}^{+\infty} \frac{\varphi(x+y) - \varphi(x)}{|y|^{1+\alpha}} \left(\kappa_-^\delta \mathbf{1}_{(-\infty, 0)}(y) + \kappa_+^\delta \mathbf{1}_{(0, +\infty)}(y) \right) dy,$$

where κ_-^δ and κ_+^δ are two non-negative constants satisfying $\kappa_-^\delta + \kappa_+^\delta > 0$ and φ is a smooth function for which the integral exists, and φ' is its derivative. This representation identifies it as the infinitesimal generator for a non-symmetric α -stable Lévy process.

Let $G_{\alpha,\delta}(t, x)$ denotes the fundamental solution of the equation $Eq_{\delta,1}^\alpha(1, 0, 0)$ that is, the unique solution of the Cauchy problem

$$\begin{cases} \frac{\partial u}{\partial t}(t, x) = D_\delta^\alpha u(t, x), \\ u(0, x) = \delta_0(x), \quad t > 0, x \in \mathbb{R}, \end{cases}$$

where δ_0 is the Dirac distribution. Using Fourier's calculus one get

$$G_{\alpha,\delta}(t, x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \exp \left(-\imath zx - t|z|^\alpha \exp \left(-\imath \delta \frac{\pi}{2} \operatorname{sgn}(z) \right) \right) dz.$$

The relevant parameters, α , called the index of *stability* and δ (related to the asymmetry) improperly referred to as the *skewness* are real numbers satisfying $\alpha \in]0, 2]$ and $|\delta| \leq \min(\alpha, 2 - \alpha)$.

The function $G_{\alpha,\delta}(t, \cdot)$ has the following properties (see e.g. [15, 16]).

Lemma 1.1 *For $\alpha \in (0, 2] \setminus \{1\}$ such that $|\delta| \leq \min(\alpha, 2 - \alpha)$*

- (i) *The function $G_{\alpha,\delta}(t, \cdot)$ is the density of a Lévy α -stable process in time t .*
- (ii) *The function $G_{\alpha,\delta}(t, x)$ is not in general symmetric relatively to x .*
- (iii) *Semi-group property: $G_{\alpha,\delta}(t, x)$ satisfies the Chapman Kolmogorov equation, i.e. for $0 < s < t$*

$$G_{\alpha,\delta}(t + s, x) = \int_{-\infty}^{+\infty} G_{\alpha,\delta}(t, y) G_{\alpha,\delta}(s, x - y) dy.$$

- (iv) *Scaling property: $G_{\alpha,\delta}(t, x) = t^{-1/\alpha} G_{\alpha,\delta}(1, t^{-1/\alpha} x)$.*

- (v) *There exists a constant c_α such that $0 \leq G_{\alpha,\delta}(1, x) \leq c_\alpha / (1 + |x|^{1+\alpha})$, for all $x \in \mathbb{R}$.*

Now, for higher dimension $d \geq 1$ and any multi index $\underline{\alpha} = (\alpha_1, \dots, \alpha_d)$ and $\underline{\delta} = (\delta_1, \dots, \delta_d)$, let $\mathbf{G}_{\underline{\alpha}, \underline{\delta}}(t, x)$ be the Green function of the deterministic equation $Eq_{\underline{\delta}, 1}^{\underline{\alpha}}(d, 0, 0)$. Clearly

$$\begin{aligned} \mathbf{G}_{\underline{\alpha}, \underline{\delta}}(t, x) &= \prod_{i=1}^d G_{\alpha_i, \delta_i}(t, x_i) \\ &= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \exp \left(-i \langle \xi, x \rangle - t \sum_{i=1}^d |\xi_i|^{\alpha_i} \exp \left(-i \delta_i \frac{\pi}{2} \text{sgn}(\xi_i) \right) \right) d\xi, \end{aligned}$$

where $\langle \cdot, \cdot \rangle$ stands for the inner product in \mathbb{R}^d .

1.2 The driving noise F and a family of SPDEs driven by F

Let us explicitly describe here the spatially homogeneous noise (see e.g. [11]). Precisely, let $\mathcal{S}(\mathbb{R}^{d+1})$ be the space of Schwartz test functions, on a complete probability space (Ω, \mathcal{G}, P) , the noise $F = \{F(\varphi), \varphi \in \mathcal{S}(\mathbb{R}^{d+1})\}$ is assumed to be an $L^2(\Omega, \mathcal{G}, P)$ -valued Gaussian process with mean zero and covariance functional given by

$$J(\varphi, \psi) := \mathbb{E}(F(\varphi)F(\psi)) = \int_{\mathbb{R}_+} \int_{\mathbb{R}^d} \left(\varphi(s, \star) * \tilde{\psi}(s, \star) \right) (x) \Gamma(dx) ds, \quad \varphi, \psi \in \mathcal{S}(\mathbb{R}^{d+1}),$$

where $\tilde{\psi}(s, x) = \psi(s, -x)$ and Γ is a non-negative and non-negative definite tempered measure, therefore symmetric. The symbols $*$ denotes the convolution product and \star stands for the spatial variable.

Let μ denotes the spectral measure of Γ (usually called the spectral measure of the noise F), which is also a trivial tempered measure (see [34, Chap. VII, Théorème XVII]), that is $\mu = \mathcal{F}^{-1}(\Gamma)$ and this gives

$$J(\varphi, \psi) = \int_{\mathbb{R}_+} ds \int_{\mathbb{R}^d} \mu(d\xi) \mathcal{F}\varphi(s, \cdot)(\xi) \overline{\mathcal{F}\psi(s, \cdot)}(\xi), \quad (1.2)$$

where \bar{z} is the complex conjugate of z .

A typical example of space correlation is given by $\Gamma(x) = f(x)dx$, where f is a nonnegative function which is assumed to be integrable around the origin. In this case, the covariance functional J reads

$$J(\varphi, \psi) = \int_{\mathbb{R}_+} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \varphi(s, x) f(x - y) \psi(s, y) dy dx ds.$$

The space-time white noise would correspond to the case where f is the Dirac delta at the origin. Following the same approach as in [11], the Gaussian process F can be extended to a worthy martingale measure $\{F(t, A) := F([0, t] \times A); t \geq 0, A \in \mathcal{B}_b(\mathbb{R}^d)\}$ which shall acts as integrator, in the sense of Walsh [38], where $\mathcal{B}_b(\mathbb{R}^d)$ denotes the bounded Borel subsets of \mathbb{R}^d . Let \mathcal{G}_t be the completion of the σ -field generated by the random variables $\{F(s, A); 0 \leq s \leq t, A \in \mathcal{B}_b(\mathbb{R}^d)\}$. The properties of F ensure that the process $F = \{F(t, A); t \geq 0, A \in \mathcal{B}_b(\mathbb{R}^d)\}$, is a martingale with respect to the filtration $\{\mathcal{G}_t; t \geq 0\}$.

Then, one can give a rigorous meaning to the solution of equation $Eq_{\delta, \varepsilon}^\alpha(d, b, \sigma)$, by means of a jointly measurable and \mathcal{G}_t -adapted process $\{u(t, x); (t, x) \in \mathbb{R}_+ \times \mathbb{R}^d\}$ satisfying, for each $t \geq 0$ and a.s. for almost all $x \in \mathbb{R}^d$ the following evolution equation:

$$\begin{aligned} u^\varepsilon(t, x) &= \sqrt{\varepsilon} \int_0^t \int_{\mathbb{R}^d} \mathbf{G}_{\alpha, \delta}(t - s, x - y) \sigma(u^\varepsilon(s, y)) F(ds, dy) \\ &\quad + \int_0^t ds \int_{\mathbb{R}^d} \mathbf{G}_{\alpha, \delta}(t - s, x - y) b(u^\varepsilon(s, y)) dy. \end{aligned} \quad (1.3)$$

Note that the last stochastic integral on the right-hand side of (1.3) can be understood in the sense of Walsh [38] or using the further extension of Dalang [11].

In order to prove our main result, we are going to give another equivalent approach to the solution of $Eq_{\delta, \varepsilon}^\alpha(d, b, \sigma)$, see [13]. To start with, let us denote by \mathcal{H} the Hilbert space obtained by the completion of $\mathcal{S}(\mathbb{R}^d)$ with the inner product

$$\langle \varphi, \psi \rangle_{\mathcal{H}} = \int_{\mathbb{R}^d} \Gamma(dx) (\varphi * \tilde{\psi})(x) = \int_{\mathbb{R}^d} \mu(d\xi) \mathcal{F}\varphi(\xi) \overline{\mathcal{F}\psi(\xi)}; \varphi, \psi \in \mathcal{S}(\mathbb{R}^d).$$

By the Walsh theory of martingale measures [38], for $t \geq 0$ and $h \in \mathcal{H}$ the stochastic integral

$$B_t(h) = \int_0^t \int_{\mathbb{R}^d} h(y) F(ds, dy),$$

is well defined and the process $\{B_t(h); t \geq 0, h \in \mathcal{H}\}$ is a cylindrical Wiener process on \mathcal{H} , that is: (a) For every $h \in \mathcal{H}$ with $\|h\|_{\mathcal{H}} = 1$, $\{B_t(h)\}_{t \geq 0}$ is a standard Wiener process; and (b) For every $t \geq 0, a, b \in \mathbb{R}$ and $f, g \in \mathcal{H}$, $B_t(af + bg) = aB_t(f) + bB_t(g)$ almost surely.

Let $(e_k)_{k \geq 1}$ be a complete orthonormal system (*CONS*) of the Hilbert space \mathcal{H} , then

$$\{B_t^k := \int_0^t \int_{\mathbb{R}^d} e_k(y) F(ds, dy); k \geq 1\}$$

defines a sequence of independent standard Wiener processes and we have the following representation

$$B_t = \sum_{k \geq 1} B_t^k e_k. \quad (1.4)$$

Let $(\mathcal{F}_t)_{t \in [0, T]}$ be the σ -field generated by the random variables $\{B_s^k; s \in [0, t], k \geq 1\}$. We define the predictable σ -field as the σ -field in $\Omega \times [0, T]$ generated by the sets $\{(s, t] \times A; A \in \mathcal{F}_s, 0 \leq s < t \leq T\}$. In the following we can define the stochastic integral with respect to cylindrical Wiener process $(B_t(h))_{t \geq 0}$ (see e.g. [14, Chapter 4] or [13]) of any predictable square-integrable process with values in \mathcal{H} as follows

$$\int_0^t \int_{\mathbb{R}^d} g \cdot dB := \sum_{k \geq 1} \int_0^t \langle g(s), e_k \rangle_{\mathcal{H}} dB_s^k.$$

Note that the above series converge in $L^2(\Omega, \mathcal{F}, P)$ and the sum does not depend on the selected *CONS*. Moreover, each summand, in the above series, is a classical Itô integral with respect to a standard Brownian motion, and the resulting stochastic integral is a real-valued random variable.

In the sequel, we shall consider the *mild* solution to equation $Eq_{\delta, \varepsilon}^{\alpha}(d, b, \sigma)$ given by

$$\begin{aligned} u^{\varepsilon}(t, x) &= \sqrt{\varepsilon} \sum_{k \geq 1} \int_0^t \langle \mathbf{G}_{\alpha, \delta}(t-s, x - \cdot) \sigma(u^{\varepsilon}(s, \cdot)), e_k \rangle_{\mathcal{H}} dB_s^k \\ &\quad + \int_0^t [\mathbf{G}_{\alpha, \delta}(t-s) * b(u^{\varepsilon}(s, \cdot))] (x) ds, \end{aligned} \quad (1.5)$$

$t \in [0, T]$, $x \in \mathbb{R}^d$ and “ $*$ ” stands for the convolution operator.

1.3 Existence, uniqueness and Hölder regularity to equation $Eq_{\delta, \varepsilon}^{\alpha}(d, b, \sigma)$

The purpose of this section is to give sufficient conditions for the existence and uniqueness to our equation and also Hölder regularity of the solution which we will use in the sequel. Now, for a given multi index $\alpha = (\alpha_1, \dots, \alpha_d)$ such that $\alpha_i \in]0, 2] \setminus \{1\}$, $i = 1, \dots, d$ and any $\xi \in \mathbb{R}^d$, let

$$S_{\alpha}(\xi) = \sum_{i=1}^d |\xi_i|^{\alpha_i}.$$

Assume the following assumptions on the functions σ , b and the measure μ :

(C) : The functions σ and b are Lipschitz.

($\mathbf{H}_{\eta}^{\alpha}$) : Let α as defined above and $\eta \in (0, 1]$

$$\int_{\mathbb{R}^d} \frac{\mu(d\xi)}{(1 + S_{\alpha}(\xi))^{\eta}} < \infty.$$

The last assumption stands for an integrability condition w.r.t. the spectral measure μ . Indeed, the following stochastic integral

$$\int_0^T \int_{\mathbb{R}^d} \mathbf{G}_{\alpha, \delta}(T-s, x-y) F(ds, dy),$$

is well defined if and only if

$$\int_0^T ds \int_{\mathbb{R}^d} \mu(d\xi) |\mathcal{F} \mathbf{G}_{\alpha, \delta}(s, \cdot)(\xi)|^2 < +\infty.$$

More precisely, appealing to [5, Lemma 1.2] there exist two positive constants c_1 and c_2 such that

$$c_1 \int_{\mathbb{R}^d} \frac{\mu(d\xi)}{1 + S_{\mathbf{Q}}(\xi)} \leq \int_0^T \int_{\mathbb{R}^d} \mu(d\xi) |\mathcal{F}\mathbf{G}_{\mathbf{Q},\delta}(s, \cdot)(\xi)|^2 ds \leq c_2 \int_{\mathbb{R}^d} \frac{\mu(d\xi)}{1 + S_{\mathbf{Q}}(\xi)}. \quad (1.6)$$

Remark 1.1 *The upper and lower bounds in (1.6) do not depend on the parameter δ . When $\alpha_i = 2$ for all $i = 1, \dots, d$ then $S_2(\xi) = \sum_{i=1}^d |\xi_i|^2 =: |\xi|^2$ and the bounds in (1.6) are the same ones which appeared in [11] (see also [33]), that is*

$$\int_{\mathbb{R}^d} \frac{\mu(d\xi)}{1 + |\xi|^2} < +\infty.$$

In the paper [9], Theorem 2.1 gives existence and uniqueness of random field solutions to equation (1.5) under conditions (C) and ($\mathbf{H}_1^{\mathbf{Q}}$) for $\eta = 1$. In fact, this result extend those of [2, Theorem 2.1] [16, Theorem 1] to the d -dimensional case and [11, Theorem 13] to the fractional setting.

Moreover, [5, Theorem 3.1] gives the regularity properties of the solution to equation (1.5) in time and space improving the results in [31, 32] corresponding to the case $\alpha_i = 2$, $\delta_i = 0$ for $i = 1, \dots, d$. More precisely, the trajectories of the solution are $\beta = (\beta_1, \beta_2)$ -Hölder continuous in $(t, x) \in [0, T] \times K$ for every $\beta_1 \in (0, \alpha_0 \frac{1-\eta}{2})$, $\beta_2 \in (0, 1 - \eta)$ and every K compact subset of \mathbb{R}^d , where $\alpha_0 = \min_{1 \leq i \leq d} \{\alpha_i\}$.

Consequently, the random field solution $\{u(t, x); (t, x) \in [0, T] \times K\}$ to equation (1.5) lives in the Hölder space $C^\beta([0, T] \times K; \mathbb{R}^d)$ equipped with the norm defined by

$$\|f\|_{\beta, K} = \sup_{(t, x) \in [0, T] \times K} |f(t, x)| + \sup_{s \neq t \in [0, T]} \sup_{x \neq y \in K} \frac{|f(t, x) - f(s, y)|}{|t - s|^{\beta_1} + \|x - y\|^{\beta_2}}. \quad (1.7)$$

Let us define the space $\mathcal{H}_T \doteq L^2([0, T]; \mathcal{H})$. For any $h \in \mathcal{H}_T$, consider the deterministic evolution equation

$$\begin{aligned} Z^h(t, x) &= \int_0^t \langle \mathbf{G}_{\mathbf{Q},\delta}(t - s, x - \cdot) \sigma(Z^h(s, \cdot)), h(s, \cdot) \rangle_{\mathcal{H}} ds \\ &\quad + \int_0^t \left[\mathbf{G}_{\mathbf{Q},\delta}(t - s) * b(Z^h(s, \cdot)) \right] (x) ds, \end{aligned} \quad (1.8)$$

where the first term on the right-hand side of the above equation can be written as

$$\sum_{k \geq 1} \int_0^t \langle \mathbf{G}_{\mathbf{Q},\delta}(t - s, x - \cdot) \sigma(Z^h(s, \cdot)), e_k \rangle_{\mathcal{H}} h_k(s) ds,$$

with $h_k(t) = \langle h(t), e_k \rangle_{\mathcal{H}}$, $t \in [0, T]$, $k \geq 1$.

For the existence and Hölderian regularities for the solution of equation (1.8) see Remark 2.6.

2 General framework and Large deviation principle result

2.1 Large deviation principle and Laplace principle

Let $\{X^\varepsilon; \varepsilon > 0\}$ be a family of random variables defined on a probability space (Ω, \mathcal{F}, P) and taking values in a Polish space (i.e. separable complete metric space) \mathcal{E} . We denote by \mathbb{E}

the expectation with respect to P . The LDP for the family $\{X^\varepsilon; \varepsilon > 0\}$ is concerned with events A for which probabilities $P(X^\varepsilon \in A)$ converges to zero exponentially fast as $\varepsilon \rightarrow 0$. The exponential decay rate of such probabilities are typically expressed in terms of a “rate function” I mapping \mathcal{E} into $[0, \infty]$.

Definition 2.1 *The family of random variables $\{X^\varepsilon; \varepsilon > 0\}$ is said to satisfy the LDP with the good rate function (or action functional) $I : \mathcal{E} \rightarrow [0, \infty]$, on \mathcal{E} , if*

1. *For each $M < \infty$ the level set $\{x \in \mathcal{E}; I(x) \leq M\}$ is a compact subset of \mathcal{E} .*
2. *Large deviation upper bound: for any closed subset F of \mathcal{E}*

$$\limsup_{\varepsilon \rightarrow 0^+} \varepsilon \log P(X^\varepsilon \in F) \leq -I(F).$$

3. *Large deviation lower bound: for any open subset O of \mathcal{E}*

$$\liminf_{\varepsilon \rightarrow 0^+} \varepsilon \log P(X^\varepsilon \in O) \geq -I(O).$$

Where, for $A \subset \mathcal{E}$, we define $I(A) = \inf_{x \in A} I(x)$.

Varadhan’s and Bryc’s results, [37] and [6], announced an equivalence between LDP and Laplace principle (LP), which notices the expectations of exponential functions.

Definition 2.2 *(Laplace principle) The family of random variables $\{X^\varepsilon; \varepsilon > 0\}$ defined on the Polish space \mathcal{E} , is said to satisfy the Laplace principle with the rate function I if for any bounded continuous function $h : \mathcal{E} \rightarrow \mathbb{R}$,*

$$\lim_{\varepsilon \rightarrow 0} \varepsilon \log \mathbb{E} \left(\exp \left[-\frac{1}{\varepsilon} h(X^\varepsilon) \right] \right) = - \inf_{f \in \mathcal{E}} \{h(f) + I(f)\}.$$

Another display of variational representation in evaluating the exponential integrals is in the following proposition which is a cornerstone of weak convergence method. For a comprehensive introduction to the applications of weak convergence method to the theory of large deviations we refer the reader to the paper [20].

Proposition 2.1 *Let $(\mathcal{V}, \mathcal{A})$ be a measurable space and f be a bounded measurable function mapping \mathcal{V} into the real numbers \mathbb{R} . For a given probability measure θ on \mathcal{V} , we have the following representation*

$$-\log \int_{\mathcal{V}} e^{-f} d\theta = \inf_{\gamma \in \mathcal{P}(\mathcal{V})} \{R(\gamma||\theta) + \int_{\mathcal{V}} f d\gamma\},$$

where $R(\gamma||\theta) := \int_{\mathcal{V}} \log(\frac{d\gamma}{d\theta}) d\gamma$ and $\mathcal{P}(\mathcal{V})$ denotes the set of probability measures on \mathcal{V} .

By using the above Proposition, the following variational representation has been obtained in [8, 9] for exponential integrals w.r.t. the Wiener process.

Variational representation

Let $B = \{B_k(t); t \in [0, T], k \geq 1\}$ be a sequence of independent real standard Brownian motions, and notice that B is a $C([0, T]; \mathbb{R}^\infty)$ -valued random variable. Consider the Hilbert space $l_2 = \{x \equiv (x_1, x_2, \dots); x_i \in \mathbb{R} \text{ and } \sum x_i^2 < \infty\}$, and let $\mathcal{P}_2(l_2)$ be the family of all l_2 -valued predictable processes for which $\int_0^T \|\phi(s)\|_{l_2}^2 ds < \infty$ a.s., where $\|\cdot\|_{l_2}$ denotes the norm in the Hilbert space l_2 . That is, $u \in \mathcal{P}_2(l_2)$ can be written as $u = \{u_i\}_{i=1}^\infty$, $u_i \in \mathcal{P}_2(\mathbb{R})$ and $\sum_{i=1}^\infty \int_0^T |u_i(s)|^2 ds < \infty$ a.s.

Theorem 2.2 ([9, Theorem 2]). *Let g be a bounded, Borel measurable function mapping $C([0, T]; \mathbb{R}^\infty)$ into \mathbb{R} . Then*

$$-\log \mathbb{E} [\exp(-g(B))] = \inf_{u \in \mathcal{P}_2(l_2)} \mathbb{E} \left[\frac{1}{2} \int_0^T \|u(s)\|_{l_2}^2 ds + g \left(B + \int_0^\cdot u(s) ds \right) \right].$$

According to [4], whenever the functionals of interest are expressed as measurable functionals of a Wiener process, the above stated representation can be used to derive various asymptotic results of large deviations nature.

This representation together with Laplace's Principle present a different method in obtaining LDP for large class of stochastic equation driven by a Gaussian type noise, by using stochastic control and weak convergence approach for a given family $\mathcal{G}^\varepsilon(B(\cdot))$, where \mathcal{G}^ε is an appropriate family of measurable maps from the Wiener space to some Polish space and $B(\cdot)$ stands for a Hilbert space valued Wiener process (see [20]).

2.2 The main result

The aim of this work is to apply the weak convergence approach to establish LDP for the family $\{u^\varepsilon; \varepsilon \in (0, 1]\}$ given by (1.5), in the space of β -Hölder continuous $C^\beta([0, T] \times K; \mathbb{R}^d)$, with the rate function I defined below by (2.10).

As mentioned in the Introduction section, to put our problem in its obvious setting, we need a Polish space carrying the probability laws of the family $\{u^\varepsilon(t, x); \varepsilon \in (0, 1], (t, x) \in [0, T] \times \mathbb{R}^d\}$. Since $C^\beta([0, T] \times K; \mathbb{R}^d)$ is not separable, we are brought to consider the space $C^{\beta', 0}([0, T] \times K; \mathbb{R}^d)$ of Hölder continuous functions f with degree $\beta' < \beta$ such that

$$\lim_{\delta \rightarrow 0^+} \left(\sup_{|t-s|+|x-y|<\delta} \frac{|f(t, x) - f(s, y)|}{|t-s|^{\beta'_1} + \|x-y\|^{\beta'_2}} \right) = 0,$$

and $C^{\beta', 0}([0, T] \times K; \mathbb{R}^d)$ is a Polish space containing $C^\beta([0, T] \times K; \mathbb{R}^d)$.

From now on, let $\mathcal{E}^\beta := C^{\beta, 0}([0, T] \times K; \mathbb{R}^d)$ be the space of (β_1, β_2) -Hölder continuous functions in time and space respectively which we equip with the Hölder norm of degree β defined by (1.7), where $\beta = (\beta_1, \beta_2)$ and satisfying $0 < \beta_1 < \frac{\alpha_0(1-\eta)}{2}$, $0 < \beta_2 < 1 - \eta$ and $\alpha_0 = \min_{1 \leq i \leq d} \{\alpha_i\}$.

We introduce the map \mathcal{G}^0 that will be used to define the rate function in our setting, that is

$$\begin{aligned} \mathcal{G}^0 : \mathcal{H}_T &\longrightarrow \mathcal{E}^\beta \\ h &\longmapsto \mathcal{G}^0(h) = Z^h, \end{aligned} \tag{2.9}$$

where Z^h is the strong solution of the integral equation defined by (1.8).

Our aim is to prove the following.

Theorem 2.3 *Assume (C) and (\mathbf{H}_η^α) for $\eta \in (0, 1]$ and $\alpha = (\alpha_1, \dots, \alpha_d)$, $\alpha_i \in]0, 2] \setminus \{1\}$ for $i = 1, \dots, d$. Let $\{u^\varepsilon(t, x); (t, x) \in [0, T] \times \mathbb{R}^d\}$ be the solution of equation (1.5). Then, the law of the solution $\{u^\varepsilon; \varepsilon \in (0, 1]\}$ satisfies on \mathcal{E}^β , a large deviation principle with rate function*

$$I(f) = \inf_{\{h \in \mathcal{H}_T; \mathcal{G}^0(h) = f\}} \left\{ \frac{1}{2} \|h\|_{\mathcal{H}_T}^2 \right\}, \quad (2.10)$$

where $\mathcal{G}^0(\cdot)$ is defined by (2.9).

In order to prove Theorem 2.3, we adopt the weak convergence approach [20]. According to [8], the crucial step in the proof is a variational representation for some functionals of an infinite dimensional Brownian motion combined with a transfer principle via Laplace Principle based on compactness and weak convergence. The authors also states that this method can be viewed as an extended contraction principle which allows to derive a Wentzell-Freidlin type large deviation results.

Accordingly, and based on this approach, we consider a set of assumptions that will be used to ensure the validity of Theorem 2.3. Now we can formulate the following sufficient conditions of the Laplace principle (equivalently, Large deviation principle) in [8] for u^ε as $\varepsilon \rightarrow 0$.

Weak regularity

Denote by $\mathcal{A}_\mathcal{H}$ the set of predictable processes which belong to $L^2(\Omega \times [0, T]; \mathcal{H})$. For any $N > 0$, define

$$\begin{aligned} \mathcal{H}_T^N &\doteq \{h \in \mathcal{H}_T; \|h\|_{\mathcal{H}_T} \leq N\}, \\ \mathcal{A}_\mathcal{H}^N &\doteq \{u(\omega) \in \mathcal{A}_\mathcal{H}; u \in \mathcal{H}_T^N \text{ a.s.}\}, \end{aligned}$$

and we consider that \mathcal{H}_T^N is endowed with the weak topology of \mathcal{H}_T .

The sets $\mathcal{A}_\mathcal{H}$ and \mathcal{H}_T^N defined above will play a central role in the proofs of the Laplace principle. Indeed, these sets are essential in proving tightness for sequences of Hilbert space valued processes by applying Theorem 2.4 and Theorem 2.5 in [8], which we will consider later.

For $v \in \mathcal{A}_\mathcal{H}^N$, $\varepsilon \in (0, 1]$ define the controlled equation $u^{\varepsilon, v}$ by

$$\begin{aligned} u^{\varepsilon, v}(t, x) &= \sqrt{\varepsilon} \sum_{k \geq 1} \int_0^t \langle \mathbf{G}_{\alpha, \delta}(t-s, x-\cdot) \sigma(u^{\varepsilon, v}(s, \cdot)), e_k \rangle_{\mathcal{H}} dB_s^k \\ &\quad + \int_0^t \langle \mathbf{G}_{\alpha, \delta}(t-s, x-\cdot) \sigma(u^{\varepsilon, v}(s, \cdot)), v(s, \cdot) \rangle_{\mathcal{H}} ds \\ &\quad + \int_0^t \mathbf{G}_{\alpha, \delta}(t-s) * b(u^{\varepsilon, v}(s, \cdot))(x) ds. \end{aligned} \quad (2.11)$$

Then, let's consider the following two conditions which correspond to the weak convergence approach framework in our setting

- a) The set $\{Z^h; h \in \mathcal{H}_T^N\}$ is a compact set of \mathcal{E}^β , Z^h being the solution of equation (1.8).
- b) For any family $\{v^\varepsilon; \varepsilon > 0\} \subset \mathcal{A}_{\mathcal{H}}^N$ which converges in distribution as $\varepsilon \rightarrow 0$ to $v \in \mathcal{A}_{\mathcal{H}}^N$, as \mathcal{H}_T^N -valued random variables, we have

$$\lim_{\varepsilon \rightarrow 0} u^{\varepsilon, v^\varepsilon} = Z^v$$

in distribution, as \mathcal{E}^β -valued random variables, where Z^v denotes the solution of (1.8) corresponding to the \mathcal{H}_T^N -valued random variable v (instead of a deterministic function h).

Remark 2.4 *We should at this point give a commentary about the two conditions considered above. Condition a) says that the level sets of the rate function are compact, and condition b) is a crucial assumption in the application of the weak convergence approach and is a statement of weak convergence of the family of random variables $\{u^{\varepsilon, v^\varepsilon}; \varepsilon > 0\}$ as ε goes to 0.*

Let

$$\mathcal{G}^\varepsilon : C([0, T]; \mathbb{R}^\infty) \longrightarrow \mathcal{E}^\beta, \varepsilon > 0$$

be a family of measurable maps such that $\mathcal{G}^\varepsilon(B(\cdot)) := u^\varepsilon$ (where u^ε stands for the solution to equation (1.5)).

Then, by applying [9, Theorem 6] to the above defined functional \mathcal{G}^ε and \mathcal{G}^0 given by (2.9), a verification of conditions a) and b) implies the validity of Theorem 2.3.

2.3 Proof of Theorem 2.3

Both conditions a) and b) will follow from a single continuity result. Condition a) will follow by proving the continuity of the mapping $h : \mathcal{H}_T^N \rightarrow Z^h \in \mathcal{E}^\beta$ with respect to the weak topology.

It will consist on proving that, if for $h, (h_n)_{n \geq 1} \subset \mathcal{H}_T^N$ such that for any $g \in \mathcal{H}_T$,

$$\lim_{n \rightarrow \infty} \langle h_n - h, g \rangle_{\mathcal{H}_T} = 0,$$

then,

$$\lim_{n \rightarrow \infty} \|Z^{h_n} - Z^h\|_{\beta, K} = 0 \tag{2.12}$$

For the condition b), we will use the Skorohod representation theorem to reformulate it. That is, there exist a probability space $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P})$, a sequence of independent Brownian motions $\bar{B} = (\bar{B}_k)_{k \geq 1}$ along with the corresponding filtration $(\bar{\mathcal{F}}_t)_{t \in [0, T]}$ where $\bar{\mathcal{F}}_t = \sigma\{\bar{B}_k(s); 0 \leq s \leq t, k \geq 1\}$ and also a family of $\bar{\mathcal{F}}_t$ -predictable processes $(\bar{v}^\varepsilon, \varepsilon > 0)$, \bar{v} belonging to $L^2(\bar{\Omega} \times [0, T]; \mathcal{H})$ taking values on \mathcal{H}_T^N \bar{P} a.s., such that the joint law of $(v^\varepsilon, v, B)_P$ coincides with that of $(\bar{v}^\varepsilon, \bar{v}, \bar{B})_{\bar{P}}$ and satisfying

$$\lim_{\varepsilon \rightarrow 0} \langle \bar{v}^\varepsilon - \bar{v}, g \rangle_{\mathcal{H}_T} = 0, \quad \bar{P} \text{ a.s.}, \quad g \in \mathcal{H}_T,$$

as \mathcal{H}_T^N -valued random variables.

Let $\bar{u}^{\varepsilon, \bar{v}^\varepsilon}(t, x)$ be the solution to a similar equation as (2.11) obtained by changing v into \bar{v}^ε and B_k by \bar{B}_k . Thus, verifying condition **b**) will consist on proving that for any $q \in [1, \infty[$ we have

$$\lim_{\varepsilon \rightarrow 0} \bar{\mathbb{E}} \left[\left\| \bar{u}^{\varepsilon, \bar{v}^\varepsilon} - Z^{\bar{v}} \right\|_{\beta, K}^q \right] = 0, \quad (2.13)$$

$\bar{\mathbb{E}}$ being the expectation operator on $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P})$.

In the other hand, notice that taking $\varepsilon = 0$ and substitute \bar{v} for $h \in \mathcal{A}_{\mathcal{H}}^N$ in (2.11) we obtain the deterministic evolution equation (1.8) satisfied by $Z^{\bar{v}}$. Accordingly, the convergence (2.12) will follow once (2.13) is proved. According to Lemma A1 in [3], the proof of (2.13) can be carried into two steps :

1- Estimation of the increments

$$\begin{aligned} \sup_{\varepsilon \leq 1} \bar{\mathbb{E}} \left(\left| \bar{u}^{\varepsilon, \bar{v}^\varepsilon}(t, x) - Z^{\bar{v}}(t, x) \right| - \left| \bar{u}^{\varepsilon, \bar{v}^\varepsilon}(s, y) - Z^{\bar{v}}(s, y) \right| \right)^q \\ \leq C \left[|t - s|^{\beta_1} + \|x - y\|^{\beta_2} \right]^q. \end{aligned} \quad (2.14)$$

2- Point-wise convergence

$$\lim_{\varepsilon \rightarrow 0} \bar{\mathbb{E}} \left(\left| \bar{u}^{\varepsilon, \bar{v}^\varepsilon}(t, x) - Z^{\bar{v}}(t, x) \right|^q \right) = 0, \quad (2.15)$$

where $q \in [1, \infty[$, $(t, x), (s, y) \in [0, T] \times K$.

First, we show the following proposition which stands for a statement of existence and uniqueness of the stochastic controlled equation given by (2.11)

Proposition 2.5 *Assuming (C) and (\mathbf{H}_η^α) , for $\eta \in (0, 1]$ and $\alpha = (\alpha_1, \dots, \alpha_d)$ satisfying $\alpha_i \in]0, 2] \setminus \{1\}$, $i = 1, \dots, d$. Then, there exists a unique random field solution to equation (2.11), $\{\bar{u}^{\varepsilon, \bar{v}^\varepsilon}(t, x); (t, x) \in [0, T] \times \mathbb{R}^d\}$, which satisfies*

$$\sup_{\varepsilon \leq 1} \sup_{v \in \mathcal{A}_{\mathcal{H}}^N} \sup_{(t, x) \in [0, T] \times \mathbb{R}^d} \bar{\mathbb{E}} \left[\left| \bar{u}^{\varepsilon, \bar{v}^\varepsilon}(t, x) \right|^q \right] < \infty. \quad (2.16)$$

Proof. From now on, we drop the bars in the notation for the sake of simplicity. We only sketch the steps of the proof following those of [5, Theorem 2.1], and is based on the Picard iteration scheme

$$\begin{aligned} u_{(0)}^{\varepsilon, v^\varepsilon}(t, x) &= 0 \\ u_{(n+1)}^{\varepsilon, v^\varepsilon}(t, x) &= \sqrt{\varepsilon} \sum_{k \geq 1} \int_0^t \langle \mathbf{G}_{\alpha, \delta}(t - s, x - \cdot) \sigma(u_{(n)}^{\varepsilon, v^\varepsilon}(s, \cdot)), e_k \rangle_{\mathcal{H}} dB_s^k \\ &\quad + \int_0^t \langle \mathbf{G}_{\alpha, \delta}(t - s, x - \cdot) \sigma(u_{(n)}^{\varepsilon, v^\varepsilon}(s, \cdot)), v^\varepsilon(s, \cdot) \rangle_{\mathcal{H}} ds \\ &\quad + \int_0^t G_{\alpha, \delta}(t - s) * b(u_{(n)}^{\varepsilon, v^\varepsilon}(s, \cdot))(x) ds. \end{aligned} \quad (2.17)$$

The first step is to check that the process $\{u_{(n)}^{\varepsilon, v^\varepsilon}(t, x); (t, x) \in [0, T] \times \mathbb{R}^d\}$ is well-defined and, for $q \geq 1$

$$\sup_{\varepsilon \leq 1} \sup_{v \in \mathcal{A}_{\mathcal{H}}^N} \sup_{(t, x) \in [0, T] \times \mathbb{R}^d} \mathbb{E} \left[\left| u_{(n)}^{\varepsilon, v^\varepsilon}(t, x) \right|^q \right] < \infty. \quad (2.18)$$

Then

$$\sup_{n \geq 0} \sup_{\varepsilon \leq 1} \sup_{v^\varepsilon \in \mathcal{A}_{\mathcal{H}}^N} \sup_{(t,x) \in [0,T] \times \mathbb{R}^d} \mathbb{E} \left[\left| u_{(n)}^{\varepsilon, v^\varepsilon}(t, x) \right|^q \right] < \infty, \quad (2.19)$$

that is the bound (2.18) holds uniformly with respect to n .

Secondly, for $n \geq 0$ let

$$M_n(t) := \sup_{(s,x) \in [0,t] \times \mathbb{R}^d} \mathbb{E} \left[\left| u_{(n+1)}^{\varepsilon, v^\varepsilon}(s, x) - u_{(n)}^{\varepsilon, v^\varepsilon}(s, x) \right|^q \right],$$

then, we prove that

$$M_{n+1}(t) \leq C_q \int_0^t M_n(s) (1 + \mathcal{J}(t-s)) ds. \quad (2.20)$$

where

$$\mathcal{J}(t-s) = \int_{\mathbb{R}^d} \mu(d\xi) \left| \mathcal{F} \mathbf{G}_{\mathbf{Q}, \delta}(t-s)(\xi) \right|^2. \quad (2.21)$$

Consequently, we can affirm that the sequence $\{u_{(n)}^{\varepsilon, v^\varepsilon}(t, x); n \geq 0\}$ converge in $L^q(\Omega)$, uniformly in (t, x) , to a limit $u^{\varepsilon, v^\varepsilon}(t, x)$ which satisfies equation (2.11) taking v^ε instead of v . Notice that equation (2.17) has an additional term, in comparison with equation (1.5) which is given by the path-wise integral

$$\int_0^t \langle \mathbf{G}_{\mathbf{Q}, \delta}(t-s, x - \cdot) \sigma(u^{\varepsilon, v^\varepsilon}(s, \cdot)), v^\varepsilon(s, \cdot) \rangle_{\mathcal{H}} ds,$$

however, the estimates (2.18), (2.19), (2.20) holds true and we proceed as follow.

As in [29, Remark 2.2], $L^q(\Omega)$ estimates of the the first and second terms in the right hand side of equation (2.11) leads, up to a constant, to the same upper bound. Indeed, since $\|v^\varepsilon\|_{\mathcal{H}_T} \leq N$ a.s., Cauchy-Schwartz's inequality on the Hilbert space \mathcal{H}_T yields, for $q \geq 1$

$$\begin{aligned} \mathbb{E} \left| \int_0^t \langle \mathbf{G}_{\mathbf{Q}, \delta}(t-s, x - \cdot) \sigma(u^{\varepsilon, v^\varepsilon}(s, \cdot)), v^\varepsilon(s, \cdot) \rangle_{\mathcal{H}} ds \right|^q \\ \leq N^q \cdot \mathbb{E} \left| \int_0^t \left\| \mathbf{G}_{\mathbf{Q}, \delta}(t-s, x - \cdot) \sigma(u^{\varepsilon, v^\varepsilon}(s, \cdot)) \right\|_{\mathcal{H}}^2 ds \right|^{\frac{q}{2}}, \end{aligned}$$

Now, by using Burkholder's inequality to the stochastic integral we obtain

$$\begin{aligned} \mathbb{E} \left| \int_0^t \langle \mathbf{G}_{\mathbf{Q}, \delta}(t-s, x - \cdot) \sigma(u^{\varepsilon, v^\varepsilon}(s, \cdot)), e_k \rangle_{\mathcal{H}} dB_s^k \right|^q \\ \leq C \cdot \mathbb{E} \left| \int_0^t \left\| \mathbf{G}_{\mathbf{Q}, \delta}(t-s, x - \cdot) \sigma(u^{\varepsilon, v^\varepsilon}(s, \cdot)) \right\|_{\mathcal{H}}^2 ds \right|^{\frac{q}{2}}, \end{aligned}$$

and this yields the extension to Theorem 2.1 in [5] to cover equation (2.11). \square

Remark 2.6 *The question of existence and uniqueness to the deterministic evolution equation defined by (1.8) will be a straightforward consequence of the last Proposition taking $\varepsilon = 0$ in (2.11).*

The next proposition is devoted to check the Hölder regularity of the stochastic integral with respect to the martingale measure F . For the proof we refer the reader to Proposition 3.2 in [5].

Proposition 2.7 *Let $\{U_{\varepsilon, v^\varepsilon}(t, x); (t, x) \in [0, T] \times \mathbb{R}^d\}$ be the stochastic integral with respect to the martingale measure F given by*

$$U_{\varepsilon, v^\varepsilon}(t, x) = \sum_{k \geq 1} \int_0^t \langle \mathbf{G}_{\mathbf{a}, \mathbf{d}}(t-s, x-\cdot) \sigma(u^{\varepsilon, v^\varepsilon}(s, \cdot)), e_k \rangle_{\mathcal{H}} dB_s^k.$$

Then, under (C) and (\mathbf{H}_η^α) , $\eta \in (0, 1]$, we have

- i) For each $x \in \mathbb{R}^d$, a.s. $t \mapsto U_{\varepsilon, v^\varepsilon}(t, x)$ is β_1 -Hölder continuous for $\beta_1 \in (0, \frac{\alpha_0(1-\eta)}{2})$,*
- ii) For each $t \in [0, T]$, a.s. $x \mapsto U_{\varepsilon, v^\varepsilon}(t, x)$ is β_2 -Hölder continuous for $\beta_2 \in (0, \min\{1 - \eta, \frac{1}{2}\})$.*

where $\alpha_0 = \min_{1 \leq i \leq d} \{\alpha_i\}$.

Now, we give the Hölder regularity to the controlled equation (2.11).

Proposition 2.8 *Assume that (C) and (\mathbf{H}_η^α) , $\eta \in (0, 1]$, holds and let $u^{\varepsilon, v^\varepsilon}$ be the solution to equation (2.11). Then $u^{\varepsilon, v^\varepsilon}$ belongs a.s. to the space \mathcal{E}^β of (β_1, β_2) -Hölder continuous functions in time and space respectively. That is, for $(t', x') \neq (t, x) \in [0, T] \times K$*

$$\mathbb{E} \left(|u^{\varepsilon, v^\varepsilon}(t', x') - u^{\varepsilon, v^\varepsilon}(t, x)|^q \right) \leq C_q \left[|t' - t|^{q \cdot \beta_1} + \|x' - x\|^{q \cdot \beta_2} \right], \quad (2.22)$$

K being a compact subset of \mathbb{R}^d . Moreover, for any $q \in [2, \infty[$

$$\sup_{\varepsilon \leq 1} \sup_{v^\varepsilon \in \mathcal{A}_{\mathcal{H}}^N} \mathbb{E} \|u^{\varepsilon, v^\varepsilon}\|_{\beta, K}^q < \infty. \quad (2.23)$$

Proof. For any $(t', x'), (t, x) \in [0, T] \times K$ such that $t' \neq t$ and $x' \neq x$, and for $q \in [2, \infty[$, consider $u^{\varepsilon, v^\varepsilon}$ and Z^v the solution to equations (2.11) and (1.8) respectively, then we have

$$\begin{aligned} \mathbb{E} \left(|[u^{\varepsilon, v^\varepsilon}(t', x') - Z^v(t', x')] - [u^{\varepsilon, v^\varepsilon}(t, x) - Z^v(t, x)]|^q \right) &\leq C_q \mathbb{E} \left(|u^{\varepsilon, v^\varepsilon}(t', x') - u^{\varepsilon, v^\varepsilon}(t, x)|^q \right) \\ &\quad + C_q \mathbb{E} \left(|Z^v(t', x') - Z^v(t, x)|^q \right). \end{aligned}$$

Thus, the estimates on the increments (2.14) will be a consequence of (2.22) since Z^v is a particular case of $u^{\varepsilon, v^\varepsilon}$ taking $\varepsilon = 0$ in equation (2.11). Now, let us focus on proving (2.22).

$$\begin{aligned}
\mathbb{E} \left(|u^{\varepsilon, v^\varepsilon}(t', x') - u^{\varepsilon, v^\varepsilon}(t, x)|^q \right) &\leq 2^{2q-2} \mathbb{E} \left| \sqrt{\varepsilon} \sum_{k \geq 1} \int_0^{t'} \langle \mathbf{G}_{\alpha, \delta}(t' - s, x' - \cdot) \sigma(u^{\varepsilon, v^\varepsilon}(s, \cdot)), e_k \rangle_{\mathcal{H}} dB_s^k \right. \\
&\quad \left. - \sqrt{\varepsilon} \sum_{k \geq 1} \int_0^t \langle \mathbf{G}_{\alpha, \delta}(t - s, x - \cdot) \sigma(u^{\varepsilon, v^\varepsilon}(s, \cdot)), e_k \rangle_{\mathcal{H}} dB_s^k \right|^q \\
&\quad + 2^{2q-2} \mathbb{E} \left| \int_0^{t'} \langle \mathbf{G}_{\alpha, \delta}(t' - s, x' - \cdot) \sigma(u^{\varepsilon, v^\varepsilon}(s, \cdot)), v^\varepsilon(s, \cdot) \rangle_{\mathcal{H}} ds \right. \\
&\quad \left. - \int_0^t \langle \mathbf{G}_{\alpha, \delta}(t - s, x - \cdot) \sigma(u^{\varepsilon, v^\varepsilon}(s, \cdot)), v^\varepsilon(s, \cdot) \rangle_{\mathcal{H}} ds \right|^q \\
&\quad + 2^{2q-2} \mathbb{E} \left| \int_0^{t'} \int_{\mathbb{R}^d} \mathbf{G}_{\alpha, \delta}(t' - s, x' - y) b(u^{\varepsilon, v^\varepsilon}(s, y)) ds dy \right. \\
&\quad \left. - \int_0^t \int_{\mathbb{R}^d} \mathbf{G}_{\alpha, \delta}(t - s, x - y) b(u^{\varepsilon, v^\varepsilon}(s, y)) ds dy \right|^q \\
&= C_q \sum_{i=1}^3 \Lambda_i,
\end{aligned}$$

where

$$\begin{aligned}
\Lambda_1 &= \mathbb{E} \left| \sqrt{\varepsilon} \sum_{k \geq 1} \int_0^{t'} \langle \mathbf{G}_{\alpha, \delta}(t' - s, x' - \cdot) \sigma(u^{\varepsilon, v^\varepsilon}(s, \cdot)), e_k \rangle_{\mathcal{H}} dB_s^k \right. \\
&\quad \left. - \sqrt{\varepsilon} \sum_{k \geq 1} \int_0^t \langle \mathbf{G}_{\alpha, \delta}(t - s, x - \cdot) \sigma(u^{\varepsilon, v^\varepsilon}(s, \cdot)), e_k \rangle_{\mathcal{H}} dB_s^k \right|^q, \\
\Lambda_2 &= \mathbb{E} \left| \int_0^{t'} \langle \mathbf{G}_{\alpha, \delta}(t' - s, x' - \cdot) \sigma(u^{\varepsilon, v^\varepsilon}(s, \cdot)), v^\varepsilon(s, \cdot) \rangle_{\mathcal{H}} ds \right. \\
&\quad \left. - \int_0^t \langle \mathbf{G}_{\alpha, \delta}(t - s, x - \cdot) \sigma(u^{\varepsilon, v^\varepsilon}(s, \cdot)), v^\varepsilon(s, \cdot) \rangle_{\mathcal{H}} ds \right|^q,
\end{aligned}$$

and

$$\begin{aligned}
\Lambda_3 &= \mathbb{E} \left| \int_0^{t'} \int_{\mathbb{R}^d} \mathbf{G}_{\alpha, \delta}(t' - s, x' - y) b(u^{\varepsilon, v^\varepsilon}(s, y)) ds dy \right. \\
&\quad \left. - \int_0^t \int_{\mathbb{R}^d} \mathbf{G}_{\alpha, \delta}(t - s, x - y) b(u^{\varepsilon, v^\varepsilon}(s, y)) ds dy \right|^q.
\end{aligned}$$

As mentioned before in the proof of Proposition 2.5, up to a constant, Λ_1 and Λ_2 have the same upper bound, which is, from Proposition 2.7, given by

$$C_q \left[|t' - t|^{q \cdot \beta_1} + \|x' - x\|^{q \cdot \beta_2} \right].$$

Now, after a change of variable, Λ_3 becomes

$$\Lambda_3 = \mathbb{E} \left| \int_0^t \int_{\mathbb{R}^d} \mathbf{G}_{\alpha, \delta}(t-s, x-y) \times [b(u^{\varepsilon, v^\varepsilon}(s+t'-t, y+x'-x)) - b(u^{\varepsilon, v^\varepsilon}(s, y))] ds dy - \int_0^{t'-t} \int_{\mathbb{R}^d} \mathbf{G}_{\alpha, \delta}(t'-s, x'-y) b(u^{\varepsilon, v^\varepsilon}(s, y)) ds dy \right|^q,$$

and we proceed as in the proof of [5, Theorem 3.1] setting $h = t' - t$ and $z = x' - x$. That is, Hölder's inequality, assertion (i) of Lemma 1.1 along with the Lipschitz condition and linear growth property of b imply

$$\begin{aligned} \Lambda_3 &\leq C_q \int_0^t \int_{\mathbb{R}^d} \mathbf{G}_{\alpha, \delta}(t-s, x-y) \times \mathbb{E} |b(u^{\varepsilon, v^\varepsilon}(s+t'-t, y+x'-x)) - b(u^{\varepsilon, v^\varepsilon}(s, y))|^q ds dy \\ &\quad + \int_0^{t'-t} \int_{\mathbb{R}^d} \mathbf{G}_{\alpha, \delta}(t'-s, x'-y) \mathbb{E} |b(u^{\varepsilon, v^\varepsilon}(s, y))|^q ds dy \\ &\leq C_q \left[|t' - t| + \int_0^t \sup_{y \in \mathbb{R}^d} \mathbb{E} |u^{\varepsilon, v^\varepsilon}(s+t'-t, y+x'-x) - u^{\varepsilon, v^\varepsilon}(s, y)|^q ds \right]. \end{aligned}$$

Putting together all the estimates and using Gronwall's lemma, we conclude the proof by the Kolmogorov continuity criterium.

Notice that going through the arguments, we can easily get uniform estimates for $u^{\varepsilon, v^\varepsilon}$ in $\varepsilon \in]0, 1]$ and $v^\varepsilon \in \mathcal{A}_{\mathcal{H}}^N$, therefore (2.23) remain valid. \square

Proposition 2.9 *Assuming (C) and (\mathbf{H}_η^α) , $\eta \in (0, 1]$, let $\{v, v^\varepsilon; \varepsilon > 0\} \subset \mathcal{A}_{\mathcal{H}}^N$, such that P.a.s.*

$$\lim_{\varepsilon \rightarrow 0} \langle v^\varepsilon - v, g \rangle_{\mathcal{H}_T} = 0, \text{ for any } g \in \mathcal{H}_T. \quad (2.24)$$

Then, for any $(t, x) \in [0, T] \times K$, $q \in [2, \infty[$ we have

$$\lim_{\varepsilon \rightarrow 0} \mathbb{E} \left(|u^{\varepsilon, v^\varepsilon}(t, x) - Z^v(t, x)|^q \right) = 0.$$

Proof. First we need to recall the following two key ingredients

$$\int_0^t ds \int_{\mathbb{R}^d} |\mathcal{F} \mathbf{G}_{\alpha, \delta}(t-s, x-\cdot)(\xi)|^2 \mu(d\xi) < \infty, \quad (2.25)$$

and

$$\sup_{v \in \mathcal{A}_{\mathcal{H}}^N} \sup_{(t, x) \in [0, T] \times \mathbb{R}^d} \mathbb{E} [|Z^v(t, x)|^q] < \infty. \quad (2.26)$$

Since condition (\mathbf{H}_η^α) holds for $\eta \in (0, 1]$, then (2.25) follow. Now, from the fact that Z^v is the solution to the particular equation (2.11) taking $\varepsilon = 0$, then (2.26) is an immediate consequence of (2.16). Fix $q \in [2, \infty[$, then

$$\mathbb{E} (|u^{\varepsilon, v}(t, x) - Z^v(t, x)|^q) \leq C_q \sum_{i=1}^4 \mathbb{E} |A_{i, \varepsilon}(t, x)|^q,$$

where

$$\begin{aligned}
A_{1,\varepsilon}(t, x) &= \int_0^t \int_{\mathbb{R}^d} \mathbf{G}_{\alpha, \delta}(t-s, x-y) [b(u^{\varepsilon, v^\varepsilon}(s, y)) - b(Z^v(s, y))] ds dy, \\
A_{2,\varepsilon}(t, x) &= \sqrt{\varepsilon} \sum_{k \geq 1} \int_0^t \langle \mathbf{G}_{\alpha, \delta}(t-s, x-\cdot) \sigma(u^{\varepsilon, v^\varepsilon}(s, \cdot)), e_k \rangle_{\mathcal{H}} dB_s^k, \\
A_{3,\varepsilon}(t, x) &= \int_0^t \langle \mathbf{G}_{\alpha, \delta}(t-s, x-\cdot) [\sigma(u^{\varepsilon, v^\varepsilon}(s, \cdot)) - \sigma(Z^v(s, \cdot))] , v^\varepsilon(s, \cdot) \rangle_{\mathcal{H}} ds, \\
A_{4,\varepsilon}(t, x) &= \int_0^t \langle \mathbf{G}_{\alpha, \delta}(t-s, x-\cdot) \sigma(Z^v(s, \cdot)), v^\varepsilon(s, \cdot) - v(s, \cdot) \rangle_{\mathcal{H}} ds.
\end{aligned}$$

For the first term $A_{1,\varepsilon}$, by Hölder's inequality along with the Lipschitz condition on b we get

$$\begin{aligned}
\mathbb{E} |A_{1,\varepsilon}(t, x)|^q &\leq \int_0^t \int_{\mathbb{R}^d} \mathbf{G}_{\alpha, \delta}(t-s, x-\cdot) \mathbb{E} |b(u^{\varepsilon, v^\varepsilon}(s, y)) - b(Z^v(s, y))|^q ds dy \\
&\leq C_q \int_0^t \sup_{(r, y) \in [0, s] \times \mathbb{R}^d} \mathbb{E} |u^{\varepsilon, v^\varepsilon}(r, y) - Z^v(r, y)|^q ds.
\end{aligned}$$

For the second term $A_{2,\varepsilon}$, Burkholder's inequality together with the linear growth property of σ and (2.26) yields

$$\begin{aligned}
\mathbb{E} |A_{2,\varepsilon}(t, x)|^q &= \varepsilon^{\frac{q}{2}} \mathbb{E} \left(\int_0^t \|\mathbf{G}_{\alpha, \delta}(t-s, x-\cdot) \sigma(u^{\varepsilon, v^\varepsilon}(s, \cdot))\|_{\mathcal{H}}^2 ds \right)^{\frac{q}{2}} \\
&\leq \varepsilon^{\frac{q}{2}} \int_0^t ds \left(1 + \sup_{(r, y) \in [0, s] \times \mathbb{R}^d} \mathbb{E} |u^{\varepsilon, v^\varepsilon}(r, y)|^q \right) \times (\mathcal{J}(t-s)) \\
&\quad \times \left(\int_0^t ds \int_{\mathbb{R}^d} |\mathcal{F} \mathbf{G}_{\alpha, \delta}(t-s, x-\cdot)(\xi)|^2 \mu(d\xi) \right)^{\frac{q}{2}-1} \\
&\leq C_q \varepsilon^{\frac{q}{2}},
\end{aligned}$$

with $\mathcal{J}(t-s)$ given by (2.21).

Then, we have

$$\lim_{\varepsilon \rightarrow 0} \mathbb{E} |A_{2,\varepsilon}(t, x)|^q = 0.$$

To deal with the term $A_{3,\varepsilon}$, first we apply the Cauchy-Schwartz's inequality to the inner product on \mathcal{H} , and the property $\sup_{\varepsilon \leq 1} \|v^\varepsilon\|_{\mathcal{H}_T} \leq N$ we have

$$\begin{aligned}
\mathbb{E} |A_{3,\varepsilon}(t, x)|^q &\leq \mathbb{E} \left(\int_0^t \|\mathbf{G}_{\alpha, \delta}(t-s, x-\cdot) [\sigma(u^{\varepsilon, v^\varepsilon}(s, \cdot)) - \sigma(Z^v(s, \cdot))]\|_{\mathcal{H}}^2 ds \right)^{\frac{q}{2}} \\
&\quad \times \left(\int_0^t \|v^\varepsilon(s, \cdot)\|_{\mathcal{H}}^2 ds \right)^{\frac{q}{2}} \\
&\leq C_q \mathbb{E} \left(\int_0^t \|\mathbf{G}_{\alpha, \delta}(t-s, x-\cdot) [\sigma(u^{\varepsilon, v^\varepsilon}(s, \cdot)) - \sigma(Z^v(s, \cdot))]\|_{\mathcal{H}}^2 ds \right)^{\frac{q}{2}}.
\end{aligned}$$

Next, Hölder's inequality with respect to the measure on $[0, T] \times \mathbb{R}^d$ given by $|\mathcal{F}\mathbf{G}_{\mathbf{Q}, \hat{\mathbf{Q}}}(t-s)(\xi)|^2 \mu(d\xi) ds$, (2.25) and the Lipschitz condition on σ yields

$$\begin{aligned} \mathbb{E} |A_{3,\varepsilon}(t, x)|^q &\leq C_q \left(\int_0^t ds \int_{\mathbb{R}^d} |\mathcal{F}\mathbf{G}_{\mathbf{Q}, \hat{\mathbf{Q}}}(t-s, x-\cdot)(\xi)|^2 \mu(d\xi) \right)^{\frac{q}{2}-1} \\ &\quad \times \int_0^t \sup_{(r,y) \in [0,s] \times \mathbb{R}^d} \mathbb{E} |u^{\varepsilon, v^\varepsilon}(r, y) - Z^v(r, y)|^q (\mathcal{J}(t-s)) ds \\ &\leq C_q \int_0^t \sup_{(r,y) \in [0,s] \times \mathbb{R}^d} \mathbb{E} |u^{\varepsilon, v^\varepsilon}(r, y) - Z^v(r, y)|^q (\mathcal{J}(t-s)) ds, \end{aligned}$$

where $\mathcal{J}(t-s)$ is defined by (2.21).

For the last term, we apply Cauchy-Schwartz's inequality to the inner product on \mathcal{H} , Hölder's inequality with respect to the measure $|\mathcal{F}\mathbf{G}_{\mathbf{Q}, \hat{\mathbf{Q}}}(t-s)(\xi)|^2 \mu(d\xi) ds$ on $[0, T] \times \mathbb{R}^d$, the linear growth property of σ and (2.26), then we get

$$\begin{aligned} \mathbb{E} |A_{4,\varepsilon}(t, x)|^q &\leq \mathbb{E} \left(\int_0^t \|\mathbf{G}_{\mathbf{Q}, \hat{\mathbf{Q}}}(t-s, x-\cdot) \sigma(Z^v(s, \cdot))\|_{\mathcal{H}}^2 ds \right)^{\frac{q}{2}} \\ &\quad \times \left(\int_0^t \|v^\varepsilon(s, \cdot) - v(s, \cdot)\|_{\mathcal{H}}^2 ds \right)^{\frac{q}{2}} \\ &\leq C_q \int_0^t ds \left(1 + \sup_{(r,y) \in [0,s] \times \mathbb{R}^d} \mathbb{E} |Z^v(r, y)|^q \right) \times (\mathcal{J}(t-s)) \\ &\quad \times \left(\int_0^t ds \int_{\mathbb{R}^d} |\mathcal{F}\mathbf{G}_{\mathbf{Q}, \hat{\mathbf{Q}}}(t-s, x-\cdot)(\xi)|^2 \mu(d\xi) \right)^{\frac{q}{2}-1} \\ &\quad \times \left(\int_0^t \|v^\varepsilon(s, \cdot) - v(s, \cdot)\|_{\mathcal{H}}^2 ds \right)^{\frac{q}{2}} \\ &\leq C_q \|v^\varepsilon - v\|_{\mathcal{H}_T}^q. \end{aligned}$$

Thus, as ε goes to 0, (2.24) imply

$$\lim_{\varepsilon \rightarrow 0} \mathbb{E} |A_{4,\varepsilon}(t, x)|^q = 0.$$

Now, let

$$\Phi_\varepsilon(t) = \sup_{(t,x) \in [0,T] \times \mathbb{R}^d} \mathbb{E} \left(|u^{\varepsilon, v^\varepsilon}(t, x) - Z^v(t, x)|^q \right).$$

Then, taking together all the estimates, we get

$$\Phi_\varepsilon(t) \leq C_q \left[\varepsilon^{\frac{q}{2}} + \mathbb{E} |A_{4,\varepsilon}(t, x)|^q + \int_0^t \Phi_\varepsilon(s) (1 + \mathcal{J}(t-s)) ds \right],$$

$\mathcal{J}(t-s)$ defined by (2.21).

We end the proof by applying the extended version of Gronwall's Lemma in [11, Lemma 15], and we find

$$\lim_{\varepsilon \rightarrow 0} \sup_{(t,x) \in [0,T] \times \mathbb{R}^d} \mathbb{E} \left(|u^{\varepsilon, v^\varepsilon}(t, x) - Z^v(t, x)|^q \right) = 0.$$

□

Proof.(of Theorem 2.3)

By Proposition 2.8 and Proposition 2.9, the estimation on increments (2.14) and Point-wise convergence (2.15) holds true. Then, as it has been argued before, Theorem 2.3 will follow. □

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